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NUMERICAL SIMULATION OF THE NEUTRAL BOUNDARY LAYER: A COMPARISON OF ENSTROPHY CONSERVING WITH MOMENTUM CONSERVING FINITE DIFFERENCE SCHEMES

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1. INTRODUCTION

The application of integral constraints to a numerically integrated system of dynamical equations is a method by which one can insure that a chaotic flow maintains its integrity. Such constraints have been a mainstay of climate models, largely out of necessity to eliminate unrealistic long term trends in such quantities as enstrophy, entropy, mass, momentum, energy, moisture and so on. It is interesting that although large eddy simulations (LESs) deal with a similar long-term integration problem (i.e. simulations over many eddy lifetimes), there has been little attention paid to the integrity of the conservation properties of the underlying dynamics schemes. This may in part be due to the strong emphasis that has been placed upon the subgrid scale diffusion schemes that are typically used to complement the dynamics schemes and in effect compensate for many of their flaws, in addition to acting as a physical representation of subgrid scale turbulence.

The Ekman boundary layer is ideal for isolating the integrity of the dynamics model of an LES because, except for the surface friction, the flow dynamics are completely inertial and do not involve the effect of stable stratification. The classic Ekman solution for the vertical structure of the horizontal velocity is derived analytically from a prescribed geostrophic wind forcing, Coriolis parameter, surface friction and the assumption of eddy mixing in the parabolic Smagorinsky form. The solution is the so-called “Ekman Spiral”. The solution is robust, i.e. not highly sensitive to surface friction parameterizations, nor to the precise formulation of the eddy-mixing coefficient.

Andren *et al.* (1994) compared the results (i.e. profiles of flux, variance, etc.) of four LES codes each simulating the classic Ekman boundary layer under a prescribed model configuration. The four schemes compared included one featuring a spectral scheme applied in the horizontal and finite difference in the vertical, and three others featuring a finite difference scheme in all three directions. None of schemes tested featured enstrophy conserving operators in all three directions. The closest of the schemes was the horizontal spectral scheme, which has no truncation on the linear part of the horizontal advection. In fact, only this scheme was run without a Galilean transformation (removal of mean velocity), most probably because of the destructive effects of truncation error present in the other schemes.

In each case, the LES simulates the Ekman layer with a combination of explicit and subgrid-scale transport. To the extent that the velocity profile is determined by the subgrid-scale transport, the simulation would be expected to reproduce the Ekman spiral. To the extent that the explicit large-eddy transport is down gradient and reproduces Smagorinsky, the explicit transport can also reproduce the Ekman Spiral.

Some numerical schemes, such as forward up-winding schemes, often have a built in implicit diffusion that may implicitly reproduce Smagorinsky mixing and push the solution towards the traditional Ekman solution. Numerical advection schemes with out diffusion added, such as second order leapfrog, have a strong tendency toward nonlinear instability over long-term integrations due to aliasing or alternatively, because of a lack of enstrophy conservation. These problems are typically controlled with the addition of subgrid-scale diffusion that acts to damp the short wavelength features and control the numerically generated enstrophy cascade. Doubling as “physical subgrid-diffusion”, these numerical corrections are indistinguishable from the real subgrid scale diffusion that exists for physical reasons. So as numerical truncation leads to enstrophy cascade, the physical diffusion resulting again mimics the classic parabolic Ekman solution, driving the solution to look more “classic” and at least appear reasonable.

This poses the question: How should a true hyperbolic Ekman solution, independent of prescribed or implied Smagorinsky mixing, look? We cannot represent an infinite series numerically nor solve the three-dimensional nonlinear problem analytically. There are, however, techniques for improving the integrity of a numerical advective scheme to conserve enstropy in the 2D limit and so reduce or eliminate the numerical enstrophy cascade. That in turn requires the parabolic destabilization techniques such as Smagorinsky diffusion or implicit diffusion built into the scheme. Because there is a “physical” enstrophy cascade in three-dimensions, the implementation of an enstrophy conserving scheme in all three vorticity planes does not eliminate the need for subgrid-scale diffusion. Rather, it tends to isolate a more physical role for subgrid-scale turbulence, consistent with the intentions of the many LES investigators that formulate these closures.

In this paper we study the performance of four advection schemes, having varying degrees of imposed integral constraints, on the simulation of the classic Ekman boundary layer. The four schemes we compare emphasize (1) momentum conservation, (2) simple enstrophy conservation for nondivergent flow, (i.e. Sadourny, 1974), and (3) enstrophy conservation for divergent flow (i.e. Arakawa and Lamb 1981) and (4) a simple second order finite difference form of the equations in advective form.

2. FORMS OF THE ADVECTION EQUATION

The second order advective form of the advection equation is, in tensor form:

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} + e_{ijk} f_j \left(u_k - u_{kg} \right) + \ldots \qquad (1)$$

The first term on the RHS is the advection term and the second term is the Coriolis term with the geostrophic base state removed. Considering finite differencing on an Arakawa "C" grid, the "advective" finite difference form of the "u" advection term is:

$$\begin{aligned} d_t \overline{u}^t &= -u d_x \overline{u}^x - \overline{v}^{x,y} d_y \overline{u}^y - \overline{w}^{x,z} d_z \overline{u}^z \\ &+ f_3 \left(\overline{v}^{x,y} - v_g \right) + \ldots \end{aligned} \qquad (2)$$

Where

$$d_x a = \frac{a_{i+1/2,\,j,k} - a_{i-1/2,\,j,k}}{x_{i+1/2} - x_{i-1/2}} \qquad (3)$$

and the averaging operator is:

$$\begin{aligned} \overline{a}^x &= \frac{a_{i+1/2\,i,\,j} + a_{i-1/2\,i,\,j}}{2}, \\ \overline{a}^{x,y} &= \frac{\overline{a}^y{}_{i+1/2\,i,\,j} + \overline{a}^y{}_{i-1/2\,i,\,j}}{2}. \end{aligned} \qquad (4)$$

Alternatively, the momentum conserving form is written:

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= -\frac{1}{r} \frac{\partial r u_j u_i}{\partial x_j} + \frac{u_i}{r} \frac{\partial r u_j}{\partial x_j} \\ &+ e_{ijk} f_j \left(u_k - u_{kg} \right) + \ldots \end{aligned} \qquad (5)$$

which has a finite difference form: :

$$\begin{aligned} & d_t \overline{u}^t = \\ & -\frac{1}{\overline{r}^x} \left(d_x \left(\overline{\overline{r}^x u}^x \, \overline{u}^x \right) + d_y \left(\overline{\left(\overline{r}^y v \right)}^x \overline{u}^y \right) + d_z \left(\overline{\left(\overline{r}^z w \right)}^x \overline{u}^z \right) \right) \\ & + \frac{u}{\overline{r}^x} \left(d_x \left(\overline{\overline{r}^x u}^x \right) + d_y \left(\overline{\left(\overline{r}^y v \right)}^x \right) + d_z \left(\overline{\left(\overline{r}^z w \right)}^x \right) \right) \\ & + f_3 \left(\overline{v}^{x,y} - v_g \right) + .. \end{aligned} \qquad (6)$$

The non-divergent enstrophy conserving form of the equation of motion can be written:

$$\frac{\partial u_i}{\partial t} = e_{ijk} h_j u_k - \frac{\partial k}{\partial x_i} - e_{ijk} f_j u_{g_k} \ldots \qquad (7)$$

where the kinetic energy, 2d potential vorticity and relative vorticity are defined:

$$\begin{aligned} h_j &= \frac{V_j + f_j}{r}, \quad V_j = e_{jkl} \frac{\partial u_k}{x_l}, \\ k &= \frac{1}{2} u_j^2 \end{aligned} \qquad (8)$$

For reference, it is interesting to note that the Ertel potential vorticity is related to h by:

$$P = h_i \frac{\partial q}{\partial x_i}. \qquad (9)$$

The 2d potential vorticity is finite differenced as:

$$h_j = \frac{\overline{\overline{e_{jkl} \left(d_{x_l} u_k \right) + f_j}^{x_j}}^{x_j}}{\overline{r}^{x_k x_l}} \; . \qquad (10)$$

Note that the vorticity is averaged twice in the direction of the vorticity vector. This has two purposes. First it defines all three 2d potential vorticity components at a unique point, being the corner of the 3d grid cube that is coincident with the 3d Ertel potential vorticity point. The final vorticity used in the calculation is then coupled into the 3d definition. Second, this couples the rotational dynamics three-dimensionally. For a two-dimensional application, this averaging has no effect.

Note also that the “j” value of vorticity is referenced by both of the $i \neq j$ component equations of velocity. Hence both component equations reference the same finite differenced evaluation of vorticity. That would not be the case for the momentum conserving form of the equations where different finite differenced vorticity fields are implied from each of the two component equations affected by vorticity. The requirement of numerical consistency is part of the basis of vorticity and enstrophy conservation.

The 2nd order enstrophy conserving finite difference forms derived by Sadourny (1975), updated by Arakawa and Lamb (1981) and applied to three dimensions by Tripoli (1991) are written here for the case of the vertical vorticity component of the “u” equation:

$$
\begin{aligned}
b_{13} &= \frac{2\overline{h_3}\rceil + b^{e}_{13}}{48} \quad ; \quad a_{13} = \frac{2\lceil\overline{h_3} + a^{e}_{13}}{48} \\
g_{13} &= \frac{2\underline{h_3}\rfloor + g^{e}_{13}}{48} \quad ; \quad d_{13} = \frac{2\lfloor\underline{h_3} + d^{e}_{13}}{48} \\
e^{b}_{13} &= d^{e}_{13} - a^{e}_{13} \quad ; \quad e^{a}_{13} = g^{e}_{13} - b^{e}_{13}
\end{aligned}
\qquad (11)
$$

where the 3 point enstrophy conserving vorticity averaging operators are defined as:

$$
\begin{aligned}
\overline{h_3}\rceil &= \left(h_{3_{i+1/2,j+1/2,k}} + h_{3_{i+1/2,j-1/2,k}} + h_{3_{i-1/2,j+1/2,k}} \right) \\
\lceil\overline{h_3} &= \left(h_{3_{i+1/2,j+1/2,k}} + h_{3_{i+1/2,j-1/2,k}} + h_{3_{i+3/2,j+1/2,k}} \right) \\
\underline{h_3}\rfloor &= \left(h_{3_{i+1/2,j+1/2,k}} + h_{3_{i+1/2,j-1/2,k}} + h_{3_{i-1/2,j-1/2,k}} \right) \\
\lfloor\underline{h_3} &= \left(h_{3_{i+1/2,j+1/2,k}} + h_{3_{i+1/2,j-1/2,k}} + h_{3_{i+3/2,j-1/2,k}} \right)
\end{aligned}
\qquad (12)
$$

The four divergence correction parameters are defined:

$$
\begin{aligned}
b^{e}_{13} &= d_{-x,-y} h_3 \quad \overline{/\|\backslash} \quad a^{e}_{13} = d_{+x,-y} h_3 \\
g^{e}_{13} &= d_{-x,+y} h_3 \quad \underline{\backslash\|/} \quad d^{e}_{13} = d_{+x,+y} h_3
\end{aligned}
\qquad (13)
$$

The bracket notation demonstrates the direction of the difference operator relative to the averaging operators given in eqs. 12. The angle bracket represents the three h positions described in eqs. (12) and the diagonal line represents the direction of the correction term described by the del operator in equation (12). These correction terms apply only to the Arakawa and Lamb (1981) technique and are zero for the Sadourny technique. These terms arise to represent the potential vorticity response to the density effects of divergence effects in the two-dimensional limit. They are particularly important in the vertical plane where density varies strongly. Analogous terms are formulated for the “y component” of vorticity in the “u equation” and for the analogous terms in the “v” and “w” equation.

A finite difference form for the kinetic energy is now required. Kinetic energy is defined at the grid center point on the “C” grid, and a simple average of the velocities across this point is possible. Arakawa and Lamb (1981) found rare instabilities could result and suggested an alternative form. Tripoli (1991) also proposed an alternative averaging form for kinetic energy. Experiments with the present Ekman test, however, revealed serious weaknesses with all of the above approaches, especially when strong mean winds are present. The weakness arises from the implicit effect on momentum conservation when the kinetic energy gradient terms of the orthogonal wind do not cancel with their opposite within the vorticity acceleration terms. The result is enstrophy conservation at the expense of momentum conservation and the spurious acceleration of flows in the vicinity of large kinetic energy gradients. In fact, in large eddy simulations, this acceleration may appear as physical role type structures, parallel to the wind. These affects were corrected by the kinetic energy averaging operator:

$$
k = \frac{1}{2}\left\{
\begin{aligned}
&+\overline{u}^{x}\,\overline{\left[\overline{\left[\overline{u}^{3y}\right]}^{3z}\right]}^{x} + \overline{v}^{y}\,\overline{\left[\overline{\left[\overline{v}^{3x}\right]}^{3z}\right]}^{y} \\
&+\overline{w}^{z}\,\overline{\left[\overline{\left[\overline{w}^{3x}\right]}^{3y}\right]}^{z}
\end{aligned}
\right\}
\qquad (14)
$$

where the “3x,3y,3z” averages are 3 point averages:

$$
\overline{a}^{3x} = \frac{a_{i-1,j,k} + a_{i,j,k} + a_{i+1,j,k}}{3} \qquad (15)
$$

From these operators, the inertial terms are finite differenced:

$$d_t \overline{u}^t = -d_x k - f_3 v_g + \begin{pmatrix} +b_{13}\overline{\overline{r}^y v}\big| & +a_{13}\big|\overline{\overline{r}^y v} \\ +g_{13}\underline{\overline{r}^y v}\big| & +d_{13}\big|\underline{\overline{r}^y v} \end{pmatrix} - \begin{pmatrix} +b_{12}\overline{\overline{r}^z w}\big| & +a_{12}\big|\overline{\overline{r}^z w} \\ +g_{12}\underline{\overline{r}^z w}\big| & +d_{12}\big|\underline{\overline{r}^z w} \end{pmatrix} \qquad (16)$$

Because of the three dimensional nature of turbulent flow, both enstrophy conserving schemes are applied separately in all three dimensions, rather than only in the vertical as in the original climate model applications.

3. EXPERIMENT DESIGN

The four numerical schemes described above were applied to the Andren *et al.* (1994) grid experimental setup. That setup studied a mean 10 m/s geostrophic wind resolved by a grid of 40 cells in each direction having 100 m spacing in the along wind direction, 50 m in the horizontal direction perpendicular to the geostrophic wind and 37.5 meters in the vertical. Turbulence intensity was set as in the Moeng design. The experiments were each initiated with a classic horizontally homogeneous wind with shear of the classic Ekman spiral and perturbed by random perturbation of intensity proportional to the shear related turbulent kinetic energy.

One of the four Andren *et al*. (1994) experiments were run with no Galilean transformation (no mean wind removed) while the others incorporated some transformation, presumably to reduce the absolute phase-error of their finite difference scheme. The case not using the transformation was a horizontally spectral model.

The present setup repeats the Andren *et al*. (1994) cases with and without the Galilean transformation. In order to eliminate grid anisotropy as a factor, the experiments were also repeated with a constant 50 m resolution in each direction. Also in order to expose the behavior of the numerical scheme, often masked by the strong role of diffusion, the Strength coefficient of the Smagorinsky diffusion was reduced to 0.5 the typical value used.

In all, 16 experiments were run.

4. RESULTS

The results suggest that the explicit transport is constantly being complimented by subgrid scale transport as one would expect. But the results also suggest the more numerical enstrophy cascade created by a scheme, the stronger the subgrid scale diffusion becomes to control the instability. Because the Ekman solution is analytically derived with a similar scheme, the mean wind profile resulting tends to look very close to the analytical profile. On the other hand, if the explicit large eddies are handled by an enstrophy conserving scheme that prevents enstrophy cascade, the subgrid scale diffusion plays a more minor role in the solution and some differences with the classic Ekman solution are noted. Most noticeable is the tendency to reduce the angle to the geostrophic wind near the surface compared to that of the Ekman solution.

The results also show a very large sensitivity of structure and numerical enstrophy cascade to the level of mean wind. These sensitivities were much worse in the case of the anisotropic grid where the along wind resolution and truncation error was reduced. The Arakawa and Lamb enstrophy-conserving operator seemed to be the most immune to these problems.

The full details of these simulations will be presented at the conference.

ACKNOWLEDGMENTS

This study was supported by NSF Grant ATM-9708314.

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